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EQUILIBRIUM CONDITIONS FOR THE 3-FACILITY HOTELLING MODEL IN A TREE*

by

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ABSTRACT

Conditions are derived under which a 3-facility Hotelling model on a tree has a locational Nash equilibrium.

Key words: Hotelling model; Nash equilibrium; minimax flow center.

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RÉSUMÉ

On dérive des conditions d'existence d'équilibre de Nash pour un modèle d'Hotelling à 3 installations sur un arbre.

Mots clés: modèle d'Hotelling; équilibre de Nash; centre de flot minimax.

1. INTRODUCTION

Since Lerner and Singer's (1937) critique of Hotelling's (1929) spatial model, it is known that the three-facility Hotelling model on a linear market has no equilibrium. De Palma et al. (1987) have shown that this result no longer holds if consumers' tastes are sufficiently heterogeneous. Eaton and Lipsey (1975) have proved that, given equal prices at all facilities, the case of three facilities is the only one for which no locational equilibrium exists. The reason for this lack of an equilibrium is based on the following consideration. On a linear market, the two peripheral facilities can both gain additional market areas by moving towards each other. This reduces the market area of the third facility, located between the other two, until it is virtually non-existent. This then forces the central facility to leap-frog to the outside, thus becoming a peripheral facility and start pushing inward itself. If the three facilities are located on a tree rather than a line, the market of the central facility is not necessarily reduced to zero so that this facility may not gain by relocating and a stable equilibrium may exist. Conditions for such an equilibrium are derived.

2. THE MODEL AND ITS PROPERTIES

Let $T = (V,E)$ be a tree with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. Let d_i denote the degree of vertex v_i and let $V(T_k)$ be the vertex set of any subtree T_k of T. A positive distance is associated with each edge E and each vertex v_i has a positive weight or demand w_i. The demand at a vertex v_i is assumed to be shared equally among the facilities closest to it. The goal is now to locate three facilities P_1, P_2 and P_3 at distinct vertices of T, so that none of the facilities can improve its market share by relocating unilaterally. Such an arrangement is called a Nash locational equilibrium. First, we will derive some properties of such an equilibrium.

Lemma 1: At equilibrium, all three facilities are located on one elementary path.

Proof: Suppose not and let the vertices be labelled so that P_i is located at v_i, i=1,2,3. Then there exists exactly one vertex v so that the paths $p_1 = (v_1, \ldots, v)$, $p_2 = (v_2, \ldots, v)$ and $p_3 = (v_3, \ldots, v)$ overlap only at v. The market areas of P_i, i=1,2,3 include the weights of all vertices v_j which can be reached from v by a path $(v, \ldots, v_i, \ldots, v_j)$. The remaining vertices on the three paths p_1, p_2 and p_3 are somehow partitioned among the three facilities. Any one of the three facilities can now gain additional market share by unilaterally moving towards v. □

Corollary 2: At equilibrium, one of the three facilities is adjacent to both remaining facilities.

Proof: Suppose not. Lemma 1 implies that there exists a path $p = (v_1, \ldots, v_2, \ldots, v_3)$ where the facilities P_1, P_2 and P_3 are located at v_1, v_2 and v_3 respectively. Then, P_1 and P_3 can both gain by moving towards P_2. □

Note that the situation described in Corollary 2 is similar to that of a linear market before the leap-frogging commences. The remaining results are based on the concept of minimax flow center on a tree.

Definition 3: A minimax flow center (MMFC) on a tree T is a point Q on T (on an edge or at a vertex) so that the largest flow of any edge in T is as small as possible, if the weights at all vertices on T are shipped on their unique path to Q.

Eiselt and Pederzoli (1984) have proved the following lemma concerning mimimax flow centers. In order to restate the lemma, define the

subtrees T_k^q, $k=1,\ldots,d_q$ generated by deleting from T vertex v_q and all of its adjacent edges. The weight $w(T_k^q)$ of a subtree is defined as the sum of all weights of the vertices of $V(T_k^q)$. The subtrees generated by v_q are then numbered so that $w(T_1^q) \geq w(T_2^q) \geq \ldots \geq w(T_{d_q}^q)$. We can now restate

Lemma 4: A minimax center of a tree T is a vertex v_q with

$$w(T_1^q) \leq \tfrac{1}{2}\, w(T) \ . \qquad (1)$$

Also note that v_q is the only MMFC of T if and only if (1) holds as a strict inequality; otherwise, there are exactly two MMFCs (Eiselt and Gendreau, 1989).

Lemma 5: At equilibrium, one of the three facilities is located at a minimax flow center.

Proof: Suppose not. According to Corollary 2, all three facilities are then located in one subtree T_k^q, with v_q being an MMFC. Thus currently, two of the facilities each have a market share smaller than $w(T_k^q) \leq w(T_1^q) \leq \frac{1}{2}w(T)$. By Lemma 4, either of these facilities could relocate to v_q, then capturing a larger market of size $w(T) - w(T_k^q) \geq w(T) - w(T_1^q) \geq \frac{1}{2}\, w(T)$. □

Assume now that facility P_i is located at vertex v_i. Then Lemma 5, together with Corollary 2, implies

Corollary 6: At equilibrium, v_1 is an MMFC and either

Case 1: $v_2 \in T_1^1$, $v_3 \in T_2^1$ and (v_1,v_2), $(v_1,v_3) \in E$ or

Case 2: v_2, $v_3 \in T_1^1$ and (v_1,v_2), $(v_2,v_3) \in E$.

These two cases can be visualized in Figures 1 and 2, respectively.

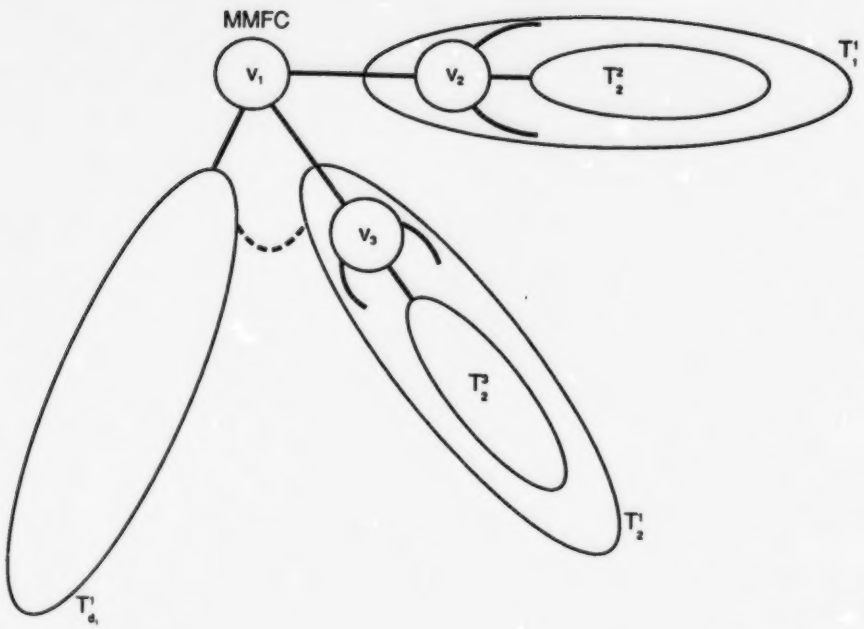

Figure 1 : Equilibrium for 3 facilities - case 1

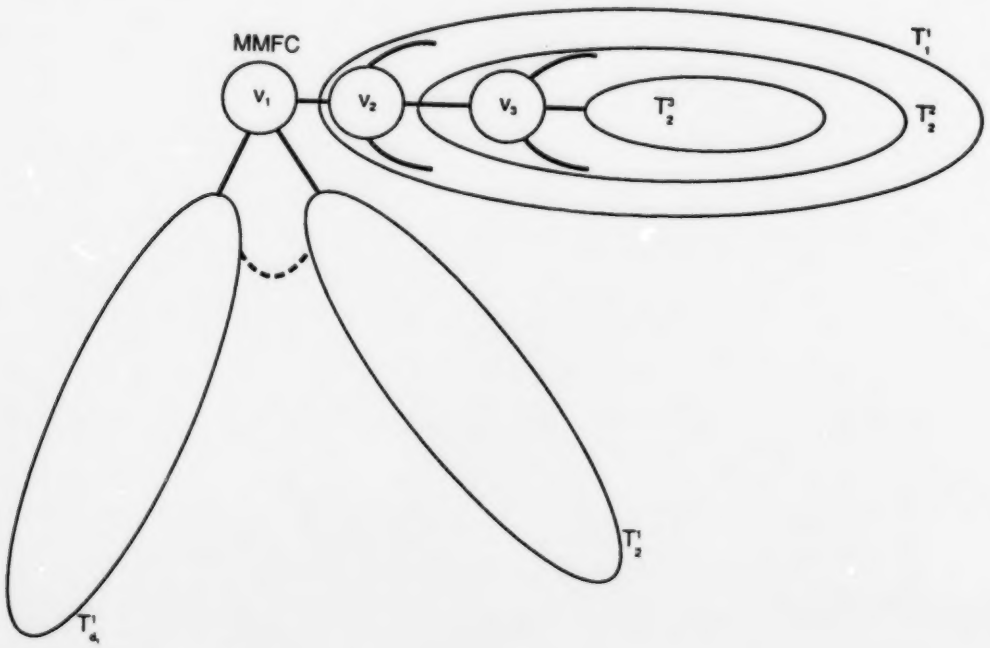

Figure 2 : Equilibrium for 3 facilities - case 2

Case 1: Currently, P_1, P_2 and P_3 have market shares of $w(T) - w(T_1^1) - w(T_2^1)$, $w(T_1^1)$ and $w(T_2^1)$, respectively. Note that $V(T_1^1) \supseteq \{v_2\} \cup V(T_2^2)$, $V(T_2^1) \supseteq \{v_3\} \cup V(T_2^3)$ and that $w(T_2^1) > w(T_2^2)$ for otherwise, P_3 would relocate, resulting in case 2. The situation described in Figure 1 is stable as long as none of the facilities can improve its market share by moving unilaterally. In particular, P_1 could possibly gain only by relocating in T_2^3, adjacent to v_2, thus capturing $w(T_2^2)$, or alternatively, in T_2^3, adjacent to v_3, thus capturing $w(T_2^3)$. Therefore, P_1 will not relocate if

$$w(T) - w(T_1^1) - w(T_2^1) > w(T_2^2) \tag{2}$$

and

$$w(T) - w(T_1^1) - w(T_2^1) > w(T_2^3) \; . \tag{3}$$

As $w(T_1^1) > w(T_k^1)$ for any $k \neq 1$, P_2 cannot improve its market share. Finally, P_3 cannot gain by relocating since $w(T_2^1) > w(T_2^2)$. Thus conditions (2) and (3) are sufficient for a stable equilibrium.

Case 2: The market shares of P_1, P_2 and P_3 are $w(T) - w(T_1^1)$, $w(T_1^1) - w(T_2^2)$ and $w(T_2^2)$, respectively. Here, $w(T_2^1) > w(T_2^2)$. Clearly, P_1 cannot improve its position by relocating. Facility P_3 could only gain by relocating in T_2^1, adjacent to v_1, provided $w(T_2^1) > w(T_2^2)$, which violates the assumption of case 2. This leaves P_2. Consider T_2^3 and observe that $V(T_2^3) \subseteq V(T_2^2) \setminus \{v_3\}$. Applying an argument similar to that of case 1, we find that P_2 will not move as long as

$$w(T_1^1) - w(T_2^2) > w(T_2^1) \tag{4}$$

and

$$w(T_1^1) - w(T_2^2) > w(T_2^3) \; . \tag{5}$$

The above two cases imply the following necessary and sufficient condition for an equilibrium.

Theorem 7: Given a tree $T = (V,E)$, and three facilities which have to be located at distinct vertices of V, a Nash locational equilibrium exists if and only if there is at least one minimax flow center v_1 and vertices v_2, v_3 satisfying:

Case 1: i) (v_1,v_2), $(v_1,v_3) \in E$;

ii) $w(T_2^1) \geq w(T_2^2)$;

iii) $w(T) - w(T_1^1) - w(T_2^1) \geq \max \{w(T_2^2), w(T_2^3)\}$;

or

Case 2: i) (v_1,v_2), $(v_2,v_3) \in E$;

ii) $w(T_2^1) \geq w(T_2^2)$;

iii) $w(T_1^1) - w(T_2^2) \geq \max \{w(T_2^1), w(T_2^3)\}$.

Note that these conditions can be checked in $O(n)$ time, the complexity of the algorithm for determining the MMFCs of a tree (Eiselt and Gendreau, 1989).

3. LOCATION ON A TREE WITH FORESIGHT

Finally consider the location of three facilities on the tree with foresight, as introduced by Prescott and Visscher (1977). In general, the first facility will locate at some vertex v_1, and the second facility will locate in T_1^1 at a vertex v_2 adjacent to v_1. The third facility then locates at v_3, which is either adjacent to v_1 in T_2^1 or adjacent to v_2 in T_2^2.

Suppose first that the third facility locates adjacent to v_2. Hence v_2 and v_3 are both in T_1^1 and it is optimal for the first facility to locate at a vertex v_q which maximizes $w(T) - w(T_1^q)$. This vertex is precisely an MMFC and the locational pattern is that depicted in Figure 2, which is an equilibrium.

Let now the third facility locate next to the first. Then P_1 locates optimally at a vertex v_q which maximizes $w(T) - w(T_1^q) - w(T_2^q)$. Such a vertex is either an MMFC or a vertex adjacent to it in its largest or second largest subtree, i.e. P_1 would locate at v_1, v_2 or v_3 in Figure 1. Facilities P_2 and P_3 will then locate at the remaining two vertices so that again an equilibrium pattern results.

In both cases, the leader (P_1) in the location game has the advantage. For a recent discussion of a model involving Stackelberg's leader-follower concept on a linear market with differential prices, the reader is referred to Anderson (1987).

BIBLIOGRAPHY

Anderson, S. (1987) "Spatial Competition and Price Leadership", International Journal of Industrial Organization 5, 369-398.

De Palma, A., V. Ginsburgh and J.-F. Thisse (1987) "On Existence of Locational Equilibria on the 3-Firm Hotelling Problem", The Journal of Industrial Economics 36, 245-252.

Eaton, B.C. and R.G. Lipsey (1975) "The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spatial Competition", Review of Economic Studies 42, 27-49.

Eiselt, H.A. and M. Gendreau (1989) "A Linear Time Algorithm for Determining the Minimax Flow Centers of a Tree", Working Paper, Centre de recherche sur les transports, Université de Montréal.

Eiselt, H.A. and G. Pederzoli (1984) "A Location Problem on Graphs", New Zealand Operational Research 12, 49-53.

Hotelling, H. (1929) "Stability in Competition", Economic Journal 39, 41-57.

Lerner, A.P. and H.W. Singer (1937) "Some Notes on Duopoly and Spatial Competition", The Journal of Political Economy 45, 145-186.

Prescott, E.C. and M. Visscher (1977) "Sequential Location Among Firms with Foresight", The Bell Journal of Economics 8, 378-393.